SYMBOLIC COMPUTING

In this edition of SYMBOLIC COMPUTING we will describe a Mercury equation file called ROCKET.EKA which can be used to determine the optimum launch mass of a rocket which maximizes the vertical altitude. This equation file was originally written for “real-time” support of model rocket altitude contests. For a given rocket engine performance capability and aerodynamic properties, there is a “best” launch mass which maximizes the altitude reached by the rocket. ROCKET.EKA is a closed-form solution to this problem.

The atmospheric density at the launch site is a function of the site altitude and temperature at the launch time as follows:

$$r = \frac{1.22557(1-2.2556913 \cdot 10^{-5} h)^{4.256116}}{1+\dfrac{(T-59)}{518.67}} \qquad (1)$$

where:

r = launch site density (kilograms/cubic meter)

h = launch site altitude (meters)

T = launch site temperature (degrees F)

The altitude at rocket engine burnout is given by:

$$X_{bo} = (m+k)\ln\left\{\cosh\left[T_d\sqrt{k(F-mg)}+m\right]\right\} \qquad (2)$$

and the burnout velocity is

$$V_{bo} = \sqrt{(F-mg)+k}\,\tanh\left\{T_d\sqrt{k(F-mg)}+m\right\} \qquad (3)$$

where:

m = average rocket mass (kilograms)

$k = \frac{1}{2} r\, C_d\, A$

r = atmospheric density

C_d = the drag coefficient of the rocket (non-dimensional)

A = cross-sectional area (square meters)

F = average thrust = $\frac{I_t}{T_d}$ (newtons)

I_t = rocket engine total impulse (newton-seconds)

T_d = thrust duration (seconds)

The altitude gained by the rocket during the coasting portion of the flight is given by:

$$X_c = (m + 2k)\ln(kV_{bo}^2 + mg + 1) \tag{4}$$

and the duration of the coasting flight is

$$T_c = \sqrt{m + kg}\ \text{atan}\left(V_{bo}\sqrt{k + mg}\right) \tag{5}$$

Finally, the maximum altitude attained by the rocket and total flight time are given by the next two “summation” equations:

$$X_{\max} = X_{bo} + X_c \tag{6a}$$

$$T_{total} = T_d + T_c \tag{6b}$$

$X_{\max}$ is the value we are trying to maximize in the following Mercury equation file.

The following is the Mercury equation file for these equations:

```
; Program "ROCKET.EKA"                 October 3, 1992

; For a given rocket engine and aerodynamic characteristics,
; this Mercury program determines the optimum launch mass of
; a single stage rocket which maximizes total altitude.

; Input

;  alt.site   = altitude at launch site          (meters)
;  temp.site  = temperature at launch site       (degrees F)
;  tburn      = thrust duration of rocket engine (seconds)
;  impulse    = total impulse of rocket engine   (newtons)
;  mprop      = rocket engine propellant mass    (kilograms)
;  diameter   = frontal diameter of rocket       (millimeters)
;  cd         = drag coefficient of rocket       (non-dimensional)

; Output

;  alt.bo   = burnout altitude            (meters)
;  vel.bo   = burnout velocity            (meters per second)
;  mass.bo  = burnout mass                (kilograms)
;  tcoast   = coast time                  (seconds)
;  tflight  = total flight time           (seconds)
;  alt.max  = maximum altitude            (meters)
;  massi    = optimum initial rocket mass (kilograms)

; maximize the total altitude with Mercury's "maximize" directive

maximize alt.max
```

```
; define the acceleration of gravity
; (meters per second per second)

gravity = 9.80665

; -----------------------------------------------------------

; NOTE: the next nine items are user inputs

; launch site altitude (meters)

alt.site = 100


; launch site temperature (degrees F)

temp.site = 70

; rocket engine thrust duration (seconds)
tburn = 1.2

; rocket engine total impulse (newton-seconds)

impulse = 5

; rocket engine propellant mass (kilograms)

mprop = .00833

; rocket frontal diameter (millimeters)

diameter = 18

; drag coefficient (non-dimensional)

cd = .321

; -----------------------------------------------------------

; constrain the initial launch mass to a value greater than
; the rocket's propellant mass (kilograms)

massi > .0083

; provide Eureka with an initial guess for the rocket's
; initial launch mass (kilograms)

; "massi" is initially set equal to the propellant mass

massi := .00833

; -----------------------------------------------------------

; compute the atmospheric density at the launch site
; (kilograms per cubic meter)

density = 1.22557*(1-2.2556913E5*alt.site)^4.256116/(1+(temp.site-
            59)/518.67)

; calculate the rocket's average thrust (newtons)

thrust = impulse / tburn

; calculate the rocket's average mass (kilograms)

mass =(massi - .5 * mprop )

; calculate the rocket's average "weight" (newtons)
```

```
weight = mass * gravity

; determine the rocket's "total" aerodynamic coefficient

k = .5*density*cd*pi*diameter^2/4e6

; compute burnout altitude (meters)

alt.bo =(mass/k)*ln(cosh(tburn*sqrt(k*(thrust-weight))/mass))

; compute burnout velocity (meters per second)

vel.bo = sqrt((thrust-weight)/k)*tanh(tburn*sqrt(k*(thrust-
            weight))/mass)

; compute burnout mass (kilograms)

mass.bo = massi - mprop

; compute burnout "weight" (newtons)

weight.bo = gravity * mass.bo

; compute coast time (seconds)

tcoast = sqrt(mass.bo/(k*gravity))*atan(vel.bo*sqrt(k/weight.bo))

; compute the total flight time of the rocket (seconds)

tflight = tburn + tcoast

; solve for the rocket's maximum altitude (meters)

alt.max = alt.bo+(.5*mass.bo/k)*ln(k*vel.bo^2/weight.bo+1)

----- Solution -----

Variables:

  alt.max  = +546.24548660164760
  gravity  = +9.80665000000000
  alt.site = +100.00000000000000
  temp.site = +70.000000000000000
  tburn    = +1.200000000000000
  impulse  = +5.0000000000000000
  mprop    = +0.00833000000000000
  diameter = +18.000000000000000
  cd       = +0.3210000000000000
  massi    = +0.0248723584909050
  density  = +1.18863832299518
  thrust   = +4.16666666666667
  mass     = +0.020707358490905044
  weight   = +0.20306981714483394
  k        = +4.8546693611572E-05
  alt.bo   = +125.11498133626417
  vel.bo   = +190.35441642504557
  mass.bo  = +0.016542358490905042
  weight.bo =+0.16222511989483396
  tcoast   = +7.5213865801020958
  tflight  = +8.7213865801020951
```

Please note that nine items must be defined by the user. These include the altitude and temperature at the launch site, the rocket engine performance, and the aerodynamic properties of the rocket's shape. The proper units for each item are noted in the equation file. The site altitude and temperature allow the software to compensate for "non-standard" conditions.

It is important that the user set the inequality constraint and the initial guess for **`massi`** equal to the propellant mass (in kilograms) of the rocket engine. For this example, the code is

```
massi > .0083

massi := .00833
```

Additional information about the algorithm used in ROCKET.EKA and other aspects of rocket flight can be found in the book *Topics in Advanced Model Rocketry* by G. J. Caporaso, G. K. Mandell and W. P. Bengen, MIT Press, 1971.