Impulsive Hyperbolic Injection from a Circular Earth Park Orbit

This document is the user's guide for a MATLAB script named `hyper1_matlab` which can be used to determine the characteristics of the single impulsive maneuver required to transfer a spacecraft from an initial circular Earth park orbit to a departure hyperbola. The algorithm implemented in this script is based on the equations derived in Chapter 4 of Richard Battin's classic text, *Astronautical Guidance*, and Chapter 11 of *An Introduction to the Mathematics and Methods of Astrodynamics*, also written by Professor Battin and published by the American Institute of Aeronautics and Astronautics (AIAA).

The Earth departure trajectory for interplanetary missions is usually defined by a "targeting specification" which consists of twice the specific (per unit mass) orbital energy C_3, and the right ascension α_∞ (RLA) and declination δ_∞ (DLA) of the outgoing asymptote. These targets may be supplied by a spacecraft customer or determined with a patched-conic or more sophisticated trajectory analysis computer program that solves Lambert's problem for an interplanetary mission.

The `hyper1_matlab` MATLAB script determines the orbital elements and state vectors of the park orbit and departure hyperbola at injection, and the injection delta-v vector and magnitude for two possible interplanetary injection opportunities. This information can be used as initial guesses for other trajectory simulations.

This computer program assumes that the hyperbolic targets and orbital characteristics are in the same Earth-centered-inertial (ECI) coordinate system. For example, targeting specs are often provided or computed in an Earth mean equator and equinox of J2000 coordinate system (EME2000). For this situation, the state vectors and orbital elements computed by this MATLAB script will also be with respect to the EME2000 coordinate system (see Appendix B).

Running the Script

The script will interactively prompt the user for the park orbit altitude and orbital inclination, and the departure hyperbola characteristics. These prompts appear as follows;

```
please input the altitude of the circular Earth park orbit (kilometers)
? 185.32

please input the orbital inclination of the Earth park orbit (degrees)
(0 <= inclination <= 180)
? 28.5

please input the C3 of the departure hyperbola (kilometers^2/second^2)
(C3 > 0)
? 9.28

please input the right ascension of the outgoing asymptote (degrees)
(0 <= right ascension <= 360)
? 352.59

please input the declination of the outgoing asymptote (degrees)
? 2.27
```

Please note the proper units and valid data range for each input.

The `hyper1_matlab` script is valid for geocentric declinations that satisfy the following geometric constraint, $|\delta_\infty| < i$ where i is the orbital inclination of the park orbit. If you define a case that violates this condition, the script will display the following message and stop.

```
****************************************************************
Please note: this script is valid for |DLA| < park orbit inclination
consider using the hyper2_matlab script for this case
****************************************************************
```

Script Output

The following is the script output for this example. For a simulation where the absolute value of the declination is equal to the park orbit inclination, the script will display a single opportunity.

```
---------------------------------------------------------
Interplanetary Injection from a Circular Earth Park Orbit
---------------------------------------------------------

departure hyperbola characteristics
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c3                             9.280000  kilometers^2/second^2

asymptote right ascension    352.590000  degrees

asymptote declination          2.270000  degrees

orbital elements and state vector of park orbit at injection - opportunity #1
------------------------------------------------------------------------------

        sma (km)            eccentricity        inclination (deg)         argper (deg)
 +6.56346000000000e+03  +0.00000000000000e+00  +2.85000000000000e+01  +0.00000000000000e+00

       raan (deg)       true anomaly (deg)         arglat (deg)          period (min)
 +1.76776733669308e+02  +2.50750239532179e+01  +2.50750239532179e+01  +8.81980522880484e+01

        rx (km)                 ry (km)                 rz (km)               rmag (km)
 -6.07292196720343e+03  -2.10640991556664e+03  +1.32727661753704e+03  +6.56346000000000e+03

        vx (kps)                vy (kps)                vz (kps)              vmag (kps)
 +2.94868471556370e+00  -6.37901947697516e+00  +3.36802611192377e+00  +7.79296034444086e+00

orbital elements and state vector of hyperbola at injection - opportunity #1
-----------------------------------------------------------------------------

        sma (km)            eccentricity        inclination (deg)         argper (deg)
 -4.29526337823278e+04  +1.15280692758590e+00  +2.85000000000000e+01  +2.50750239532179e+01

       raan (deg)         true anomaly (deg)         arglat (deg)
 +1.76776733669308e+02  +3.60000000000000e+02  +2.50750239532179e+01

        rx (km)                 ry (km)                 rz (km)               rmag (km)
 -6.07292196720343e+03  -2.10640991556664e+03  +1.32727661753704e+03  +6.56346000000000e+03

        vx (kps)                vy (kps)                vz (kps)              vmag (kps)
 +4.32644193839255e+00  -9.35958234033584e+00  +4.94171836796140e+00  +1.14341795446834e+01

injection delta-v vector and magnitude - opportunity #1
-------------------------------------------------------

x-component of delta-v       1377.757223  meters/second
y-component of delta-v      -2980.562863  meters/second
z-component of delta-v       1573.692256  meters/second

delta-v magnitude            3641.219200  meters/second
```

```
orbital elements and state vector of park orbit at injection - opportunity #2
-----------------------------------------------------------------------------

        sma (km)             eccentricity        inclination (deg)         argper (deg)
 +6.56346000000000e+03  +0.00000000000000e+00  +2.85000000000000e+01  +0.00000000000000e+00


       raan (deg)         true anomaly (deg)         arglat (deg)           period (min)
 +3.48403266330692e+02  +2.14598145959694e+02  +2.14598145959694e+02  +8.81980522880484e+01

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -5.95084600464941e+03  -2.12228648103629e+03  -1.77829668305340e+03  +6.56346000000000e+03

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 +3.20139662999697e+00  -6.41188587565448e+00  -3.06088386990660e+00  +7.79296034444086e+00

orbital elements and state vector of hyperbola at injection - opportunity #2
----------------------------------------------------------------------------

        sma (km)             eccentricity        inclination (deg)         argper (deg)
 -4.29526337823282e+04  +1.15280692758590e+00  +2.85000000000000e+01  +2.14598145959694e+02

       raan (deg)           true anomaly (deg)         arglat (deg)
 +3.48403266330692e+02  +0.00000000000000e+00  +2.14598145959694e+02

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -5.95084600464941e+03  -2.12228648103629e+03  -1.77829668305340e+03  +6.56346000000000e+03

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 +4.69723214840202e+00  -9.40780538868671e+00  -4.49106554980788e+00  +1.14341795446834e+01

injection delta-v vector and magnitude - opportunity #2
-------------------------------------------------------

x-component of delta-v       1495.835518  meters/second
y-component of delta-v      -2995.919513  meters/second
z-component of delta-v      -1430.181680  meters/second

delta-v magnitude            3641.219200  meters/second
```

Trajectory Graphics

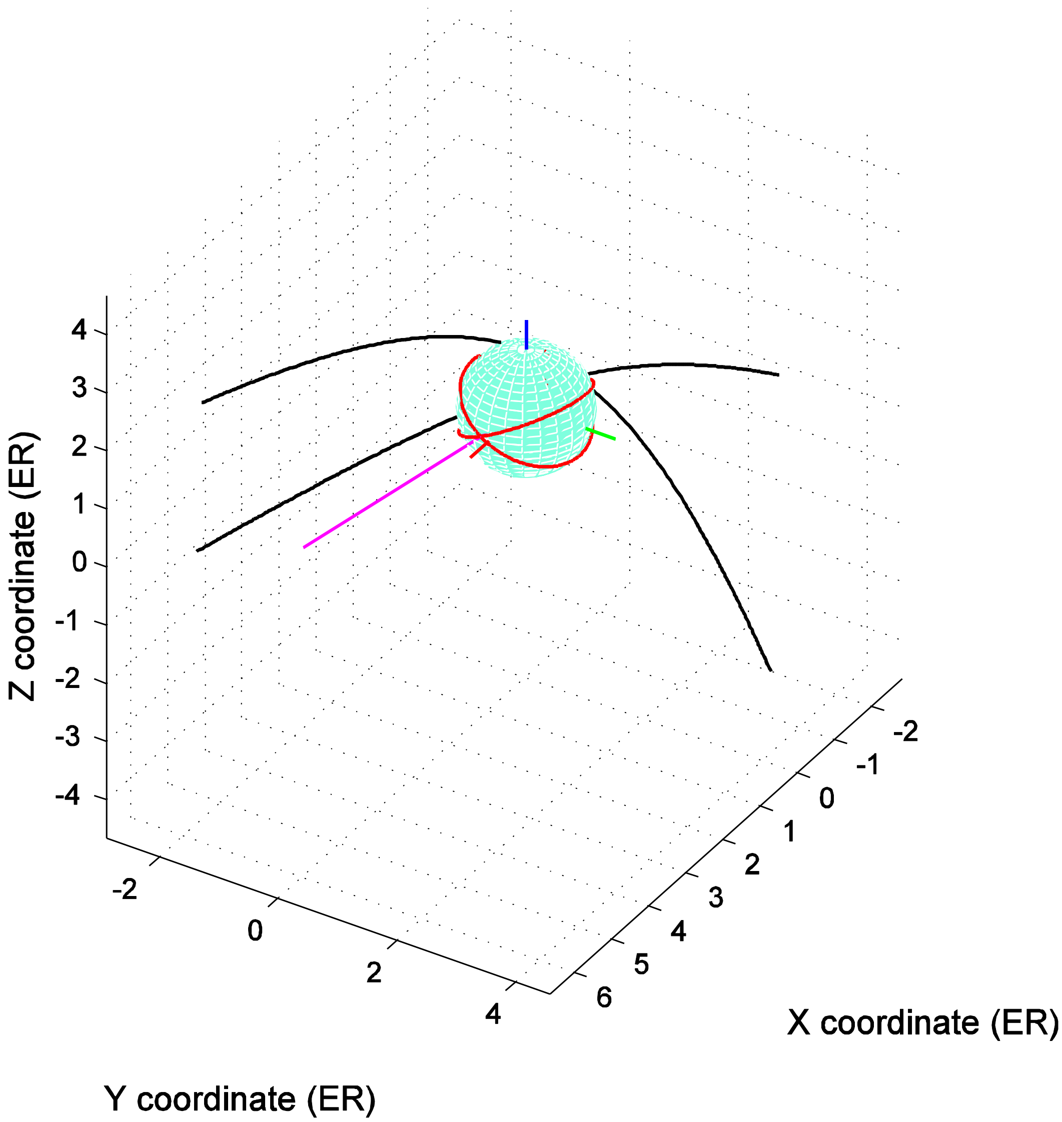
The `hyper1_matlab` MATLAB script will also create a graphics display of the park orbit and departure hyperbola for the possible opportunities. The interactive graphic features of MATLAB permit the user to rotate and zoom this display. These capabilities allow the user to interactively find the best viewpoint as well as verify basic orbital geometry of the park orbit and departure trajectory.

This script will also create a disk file of this graphics display called `hyper1_matlab.eps`. This file is a color Postscript image with a TIFF preview. The name and other characteristics of this file can be edited in the following line of MATLAB source code;

```
print -depsc -tiff -r300 hyper1_matlab.eps
```

The following is the display for this example. This display is labeled with an Earth centered, inertial coordinate system. The x-axis of this system is red, the y-axis is green and the z-axis is blue. The outgoing asymptote is colored magenta, the park orbit traces are red, and the hyperbolic trajectories are black. Please note the units for each coordinate axis are Earth radii (ER). The locations of the impulsive maneuvers are labeled on the trajectories by a small red circle.

On the graphic display, the outgoing leg of the injection hyperbolas appears to be parallel to the asymptote vector.



Technical Discussion

This section describes the numerical algorithms implemented in this MATLAB script. The script assumes that injection occurs impulsively at perigee of the departure hyperbola.

This MATLAB script is valid for geocentric orbit inclinations that satisfy the following geometric constraint

$$|\delta_\infty| < i$$

where i is the orbital inclination of the park orbit.

Whenever $|\delta_\infty| < i$, there will be two opportunities to establish a departure hyperbola that will satisfy the energy and orientation of the outgoing trajectory. Typically, one injection opportunity will occur while the spacecraft is ascending and the other while the spacecraft is descending along the park orbit. For the case where $i = |\delta_\infty|$, there will be a single injection opportunity.

A geocentric unit vector is the direction of the departure asymptote is given by

$$\hat{\mathbf{s}} = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

where

$\alpha_\infty =$ right ascension of departure asymptote

$\delta_\infty =$ declination of departure asymptote

The angle between the outgoing asymptote and the spin axis of the Earth is given by

$$\beta = \cos^{-1}\left(\hat{\mathbf{s}} \bullet \hat{\mathbf{z}}\right)$$

where $\hat{\mathbf{z}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Note that $\beta = 90^\circ - \delta_\infty$.

Departure delta-V

The velocity vector at any geocentric position vector $\mathbf{r}$ required to achieve a launch hyperbola defined by V_∞, α_∞ and δ_∞ is given by

$$\mathbf{V}_h = \left(d + \frac{1}{2}V_\infty\right)\hat{\mathbf{s}} + \left(d - \frac{1}{2}V_\infty\right)\hat{\mathbf{r}}$$

where

$$d = \sqrt{\frac{\mu}{\left(1+\cos\psi\right)r_p} + \frac{V_\infty^2}{4}}$$

and ψ is the angle between the spacecraft's unit position vector and the departure asymptote unit vector which can be computed using

$$\cos\psi = \hat{\mathbf{s}} \bullet \hat{\mathbf{r}}$$

The position vector of the spacecraft prior to and immediately after the injection maneuver consists of the following three Cartesian components;

$$r_x = r\left(\cos\Omega\cos\theta - \sin\Omega\cos i \sin\theta\right)$$

$$r_y = r\left(\sin\Omega\cos\theta + \cos\Omega\cos i \sin\theta\right)$$

$$r_z = r\sin i \sin\theta$$

where r is the geocentric radius at injection, Ω is the right ascension of the ascending node, i is the orbital inclination and θ is the true anomaly. The argument of perigee of a circular orbit is zero.

The injection $\Delta\mathbf{V}$ vector can be determined from the following expression

$$\Delta\mathbf{V} = \mathbf{V}_h - \mathbf{V}_p$$

where $\mathbf{V}_p$ is the inertial velocity vector in the park orbit prior to injection. The unit position vector is given by $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

Finally, the scalar injection delta-v is $\Delta V = |\Delta\mathbf{V}|$.

Orientation of the park orbit and departure hyperbola

This section summarizes the equations used to determine the right ascension of the ascending node (RAAN) of the Earth park orbit and the injection true anomaly on the circular park orbit.

The park orbit right ascension of the ascending node for each opportunity can be determined from

$$\Omega_1 = 180^\circ + \alpha_\infty + \sin^{-1}\left(\frac{\cot\beta}{\tan i}\right)$$

$$\Omega_2 = 360^\circ + \alpha_\infty - \sin^{-1}\left(\frac{\cot\beta}{\tan i}\right)$$

The true anomaly on the park orbit for each injection opportunity can be determined from

$$\theta_1 = \cos^{-1}\left(\frac{\cos\beta}{\sin i}\right) - \eta$$

$$\theta_2 = -\cos^{-1}\left(\frac{\cos\beta}{\sin i}\right) - \eta$$

where

$$\eta = \sin^{-1}\left(\frac{1}{1 + r_p V_\infty^2/\mu}\right)$$

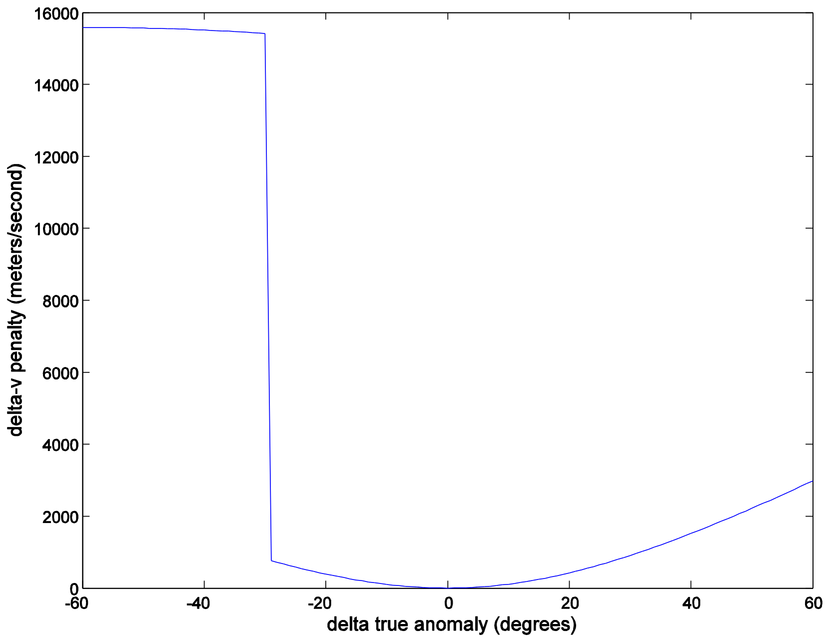
In the last equation, r_p is the geocentric radius of the circular park orbit and μ is the gravitational constant of the Earth. The speed at infinity V_∞ is determined from $V_\infty = \sqrt{C_3}$.

For an impulsive injection maneuver that occurs at perigee of the hyperbola, the true anomaly on the hyperbola is zero by definition. Furthermore, since the orbit transfer is coplanar, the right ascension of the ascending node computed above should be the same for both the Earth park orbit and the launch hyperbola. This can be verified by examining the hyperbola's RAAN which is computed by the script using the state vector at injection. The "solved for" classical orbital elements are used in the previous equations to determine the inertial position and velocity vectors of the park orbit at injection.

Delta-v penalty for off-nominal injection

The velocity-required equation given above can also be used to access the delta-v penalty for off-nominal injection. Such things as ignition timing errors and other spacecraft contingencies may result in an injection maneuver that does not occur at the optimal true anomaly on the park orbit.

The following plot illustrates how the injection delta-v penalty changes as the true anomaly at injection is displaced from the optimal.



Algorithm Resources

"Asymptotically Optimum Two-Impulse Transfer from Lunar Orbit", P. Gunther, *AIAA Journal*, Vol. 4, No. 2, February 1966.

"Optimal Noncoplanar Escape from Circular Orbits", T. N. Edelbaum, *AIAA Journal*, Vol. 9, No. 12, December 1971.

"A Simple Targeting Procedure for Lunar Trans-Earth Injection", Shane B. Robinson and David K. Geller, AIAA 2009-6107, August 2009.

"Optimal Impulsive Escape Trajectories from a Circular Orbit to a Hyperbolic Excess Velocity Vector", D. R. Jones and C. Ocampo, AIAA 2010-7524, August 2010.

An Introduction to the Mathematics and Methods of Astrodynamics, Richard H. Battin, AIAA Education Series, 1987.

Spacecraft Mission Design, Charles D. Brown, AIAA Education Series, 1992.

Orbital Mechanics, Vladimir A. Chobotov, AIAA Education Series, 2002.

Appendix A

Verification of the Required Impulsive Delta-V

P. Gunther in "Asymptotically Optimum Two-Impulse Transfer from Lunar Orbit" derives the following quartic equation for impulsive transfer from a circular park orbit to an outgoing v-infinity vector,

$$w^4 - kw^3 - k\sin^2 i_\infty w - \sin^2 i_\infty = 0$$

In this equation, $k = V_\infty / V_{lc}$, where V_∞ is the speed at "infinity", $V_{lc} = \sqrt{\mu/r}$ is the local circular speed on the Earth park orbit and *r* is the geocentric radius of the park orbit. Also, i_∞ is the (minimum) angle between the plane of the Earth park orbit and the v-infinity vector of the outgoing hyperbola.

The ratio of the magnitude of the impulsive maneuver to the local circular speed is given by

$$\frac{\Delta V}{V_{lc}} = \sqrt{k^2 + 3 - 2\sqrt{\left(1 + kw - w^2\right)\left(2 + kw\right)}}$$

We can use this technique to verify the numerical results of the algorithm used in this MATLAB script.

For a coplanar maneuver, $i_\infty = 0$, the quartic equation simplifies to

$$w^4 - kw^3 = 0 \rightarrow w = k$$

and the delta-v equation becomes

$$\frac{\Delta V}{V_{lc}} = \sqrt{k^2 + 3 - 2\sqrt{\left(1 + k^2 - k^2\right)\left(2 + k^2\right)}} = \sqrt{k^2 + 3 - 2\sqrt{2 + k^2}}$$

For the example given earlier in this document,

$$V_\infty = \sqrt{C_3} = 3046.3092 \text{ meters/second} \quad V_{lc} = 7793.032 \text{ meters/second} \quad k = 0.3909$$

Finally, substituting these values into the delta-v equation,

$$\frac{\Delta V}{V_{lc}} = 0.46724 \rightarrow \Delta V = 3641.2 \text{ meters/second}$$

which agrees with the results computed for this example by this script.

Appendix B

The EME2000 Coordinate System

The following figure illustrates the geometry of the EME2000 coordinate system. The origin of this Earth-centered-inertial (ECI) inertial coordinate system is the geocenter and the fundamental plane is the Earth's mean equator. The z-axis of this system is normal to the Earth's mean equator at epoch J2000, the x-axis is parallel to the vernal equinox of the Earth's mean orbit at epoch J2000, and the y-axis completes the right-handed coordinate system. The epoch J2000 is the Julian Date 2451545.0 which corresponds to January 1, 2000, 12 hours Terrestrial Time (TT).

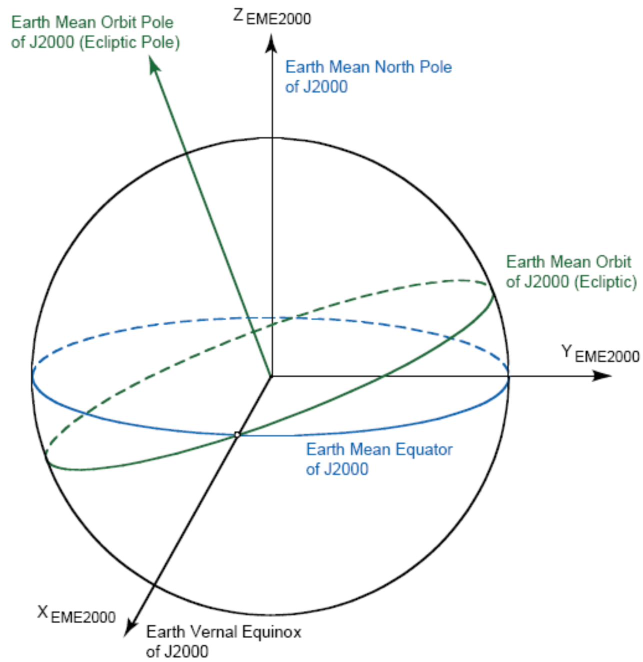


Figure 1. Earth mean equator and equinox of J2000 coordinate system