A MATLAB Script for Ballistic Interplanetary Trajectory Design and Optimization

This document describes a MATLAB script called `ipto_matlab` that can be used to design and optimize "patched conic" ballistic interplanetary trajectories between any two planets of our solar system. It can also be used to find two-body trajectories between a planet and an asteroid or comet. A patched-conic trajectory ignores the gravitational effect of both the launch and arrivals planets on the heliocentric transfer trajectory. This technique involves the solution of Lambert's problem relative to the Sun. Patched-conic trajectories are suitable for preliminary mission design. The `ipto_matlab` script also includes the option to include user-defined mission constraints such as departure energy, time-of-flight, and so forth during the trajectory optimization.

The `ipto_matlab` MATLAB script also performs a graphical primer vector analysis of the solution. This program feature displays the behavior of the primer vector magnitude and primer derivative magnitude as a function of mission elapsed time in days from departure. This script uses the SNOPT nonlinear programming (NLP) algorithm to solve this classic astrodynamics problem.

User interaction with script

The software will ask the user for an initial guess for the launch and arrival calendar dates as well as the launch and arrival celestial bodies. The script will also ask the user for a search boundary, in days, on the launch and arrival dates. The algorithm will restrict its search for the optimum launch date D_L and arrival date D_A as follows:

$$D_{L_g} - \Delta D_L \le D_L \le D_{L_g} + \Delta D_L$$

$$D_{A_g} - \Delta D_A \le D_A \le \Delta D_A + \Delta D_A$$

where D_{L_g} and D_{A_g} are the user's initial guess for launch and arrival dates, and ΔD_L, ΔD_A, are the user-specified search boundaries for the launch and arrival dates, respectively.

The following is typical user interaction with this MATLAB application. This example is an Earth-to-Mars mission that minimizes the total delta-v. The user inputs for this example are in bold font.

```
departure conditions - start date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 6,1,2003

please input the departure date search boundary in days
? 30


arrival conditions - start date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 12,1,2003

please input the arrival date search boundary in days
? 30
```

```
 celestial body menu

  <1>  Mercury
  <2>  Venus
  <3>  Earth
  <4>  Mars
  <5>  Jupiter
  <6>  Saturn
  <7>  Uranus
  <8>  Neptune
  <9>  Pluto
  <10> asteroid/comet

please select the departure celestial body
? 3

 celestial body menu

  <1>  Mercury
  <2>  Venus
  <3>  Earth
  <4>  Mars
  <5>  Jupiter
  <6>  Saturn
  <7>  Uranus
  <8>  Neptune
  <9>  Pluto
  <10> asteroid/comet

please select the arrival celestial body
? 4


would you like to enforce mission constraints (y = yes, n = no)
? n

      optimization menu

 <1> minimize departure delta-v

 <2> minimize arrival delta-v

 <3> minimize total delta-v

 <4> no optimization

 selection (1, 2 or 3)
? 3
```

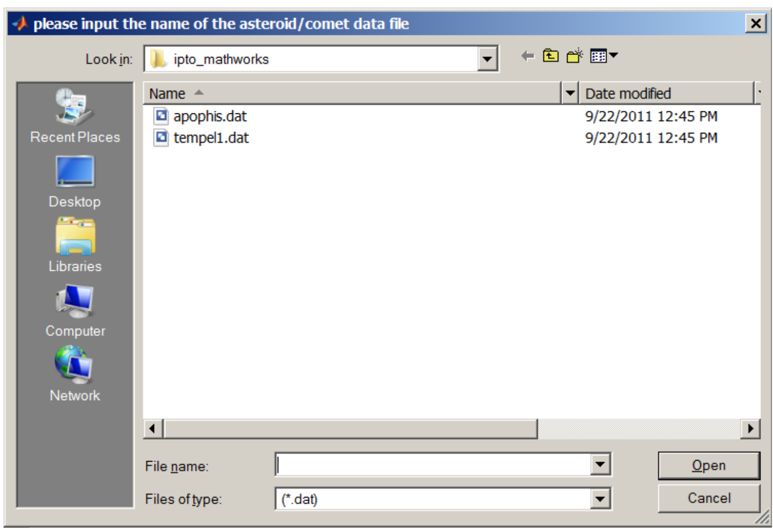
If the analyst selects an asteroid or comet as the launch or arrival body, the software will interactively prompt the user for the name of a simple data file containing the orbital elements of the object. The orbital elements of an asteroid or comet relative to the ecliptic and equinox of J2000 coordinate system must be provided by the user. The following is a typical data file for the comet Tempel 1. Do not change the number of lines of information in these data files. The data values are in bold. Please note the correct units for each orbital element. The perihelion passage calendar date should be on the Barycentric Dynamical Time (TDB) scale.

```
*******************************************
* asteroid/comet classical orbital elements *
* (heliocentric, Earth mean ecliptic J2000) *
*******************************************
```

```
asteroid/comet name
Tempel 1

TDB calendar date of perihelion passage (month, day, year)
7, 5.3153, 2005

perihelion distance (au)
1.506167

orbital eccentricity (non-dimensional)
0.517491

orbital inclination (degrees)
10.5301

argument of perihelion (degrees)
178.8390

longitude of the ascending node (degrees)
68.9734
```

Orbital elements for several comets and asteroids can be obtained from the JPL Near Earth Object (NEO) website which is located at http://neo.jpl.nasa.gov.

The prompt for the name of the asteroid or comet data file is similar to the following;



The default filename type is `*.dat`. However, the script will read any compatible data file.

If the user elects to enforce mission constraints during the solution process, the script will display the following six interactive prompts. These prompts allow the user to specify lower and upper bounds for each mission constraint. Please note the proper units for each constraint and make sure the upper bound has a larger value than the lower bound. Appendix B contains a detailed Earth-to-Mars trajectory optimization example with user-defined mission constraints.

```
please input the lower bound for departure C3 (kilometers^2/second^2)
?

please input the upper bound for departure C3 (kilometers^2/second^2)
?
```

```
please input the lower bound for departure DLA (degrees)
?

please input the upper bound for departure DLA (degrees)
?

please input the lower bound for time-of-flight (days)
?

please input the upper bound for time-of-flight (days)
?

please input the lower bound for arrival v-infinity (kilometers/second)
?

please input the upper bound for arrival v-infinity (kilometers/second)
?
```

Optimal solution and trajectory graphics display

This section summarizes the program output for this example. The information provided by the software includes the heliocentric orbital elements and state vectors of the initial orbit, transfer trajectory and final mission orbit in the J2000 mean ecliptic and equinox coordinate system. These numerical results also include the characteristics of the launch or departure hyperbola. Also note that the time scale is Barycentric Dynamical Time (TDB).

```
        program ipto_matlab

      minimize total delta-v

departure celestial body     Earth

departure calendar date      06-Jun-2003
departure TDB time           08:17:20.579

departure julian date        2452796.8454


arrival celestial body       Mars

arrival calendar date        27-Dec-2003
arrival TDB time             17:03:45.061

arrival julian date          2453001.2109

transfer time                  204.3656  days

heliocentric orbital conditions prior to the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.4963256152e+08  1.6337778758e-02  3.4017832513e-04  2.7047636439e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 1.9168183209e+02  1.5304069734e+02  6.3517061725e+01  3.6538395691e+02

        rx (km)                 ry (km)                rz (km)                rmag (km)
 -3.87803247685300e+07  -1.46766164749510e+08  +8.06715976461768e+02  +1.51803230219530e+08

        vx (kps)                vy (kps)               vz (kps)               vmag (kps)
 +2.83211858784698e+01  -7.71155136415724e+00  +7.88832797096184e-05  +2.93523013409698e+01


heliocentric orbital conditions after the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------
```

```
      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.8842763117e+08  1.9438220509e-01  7.1005826334e-02  1.7892666348e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 7.5444586386e+01  8.2764375478e-01  1.7975430724e+02  5.1632980035e+02

        rx (km)                 ry (km)                rz (km)                rmag (km)
 -3.87803247685300e+07  -1.46766164749510e+08  +8.06715976461768e+02  +1.51803230219530e+08

        vx (kps)                vy (kps)               vz (kps)               vmag (kps)
 +3.12218065445462e+01  -8.32837362979744e+00  -4.00448025834237e-02  +3.23135360926159e+01


heliocentric orbital conditions prior to the second maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.8842763117e+08  1.9438220509e-01  7.1005826334e-02  1.7892666348e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 7.5444586386e+01  1.5417471525e+02  3.3310137874e+02  5.1632980035e+02

        rx (km)                 ry (km)                rz (km)                rmag (km)
 +1.45484494989705e+08  +1.64706475639684e+08  -1.23212009645380e+05  +2.19758905578943e+08

        vx (kps)                vy (kps)               vz (kps)               vmag (kps)
 -1.52127795066753e+01  +1.64941187732280e+01  +2.33850380748383e-02  +2.24384750213574e+01


heliocentric orbital conditions after the second maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2793906443e+08  9.3540802481e-02  1.8493566928e+00  2.8651704692e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9540934984e+01  7.2487482568e+01  3.5900452948e+02  6.8697107415e+02

        rx (km)                 ry (km)                rz (km)                rmag (km)
 +1.45484494989705e+08  +1.64706475639684e+08  -1.23212009645380e+05  +2.19758905578943e+08

        vx (kps)                vy (kps)               vz (kps)               vmag (kps)
 -1.72343278293651e+01  +1.81081767751661e+01  +8.02808925319584e-01  +2.50114498584402e+01


heliocentric orbital conditions of arrival body
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2793906443e+08  9.3540802481e-02  1.8493566928e+00  2.8651704692e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9540934984e+01  7.2487482568e+01  3.5900452948e+02  6.8697107415e+02

        rx (km)                 ry (km)                rz (km)                rmag (km)
 +1.45484494989705e+08  +1.64706475639684e+08  -1.23212009645380e+05  +2.19758905578943e+08

        vx (kps)                vy (kps)               vz (kps)               vmag (kps)
 -1.72343278293651e+01  +1.81081767751661e+01  +8.02808925319584e-01  +2.50114498584402e+01


departure delta-v and energy requirements
(mean equator and equinox of J2000)
----------------------------------

x-component of delta-v       2900.620666  meters/second
y-component of delta-v       -549.963079  meters/second
z-component of delta-v       -282.170569  meters/second

delta-v magnitude            2965.751147  meters/second
```

```
energy                          8.795680  kilometers^2/second^2

asymptote right ascension     349.264051  degrees
asymptote declination          -5.459552  degrees


arrival delta-v and energy requirements
(mean equator and equinox of J2000)
-----------------------------------

x-component of delta-v       2021.548323  meters/second
y-component of delta-v      -1170.832247  meters/second
z-component of delta-v      -1357.142837  meters/second

delta-v magnitude            2701.729530  meters/second

energy                          7.299342  kilometers^2/second^2

Mars-mean-equator and IAU node of epoch

asymptote right ascension     280.631366  degrees
asymptote declination           6.277437  degrees


total delta-v                5667.480677  meters/second

total energy                   16.095022  kilometers^2/second^2
```

Special case

If the arrival planet is Mars, the software will provide the right ascension and declination of the incoming or arrival hyperbola asymptote in the Mars mean equator and IAU node of epoch coordinate system. Appendix C explains the orientation of this system and the algorithm used to compute the necessary coordinate transformation.

After the solution is displayed, the software will ask the user if he or she would like to create a graphics display of the planet orbits and transfer trajectory with the following prompt:

```
would you like to plot this trajectory (y = yes, n = no)
?
```

If the user's response is `y` for yes, the script will request a plot step size with

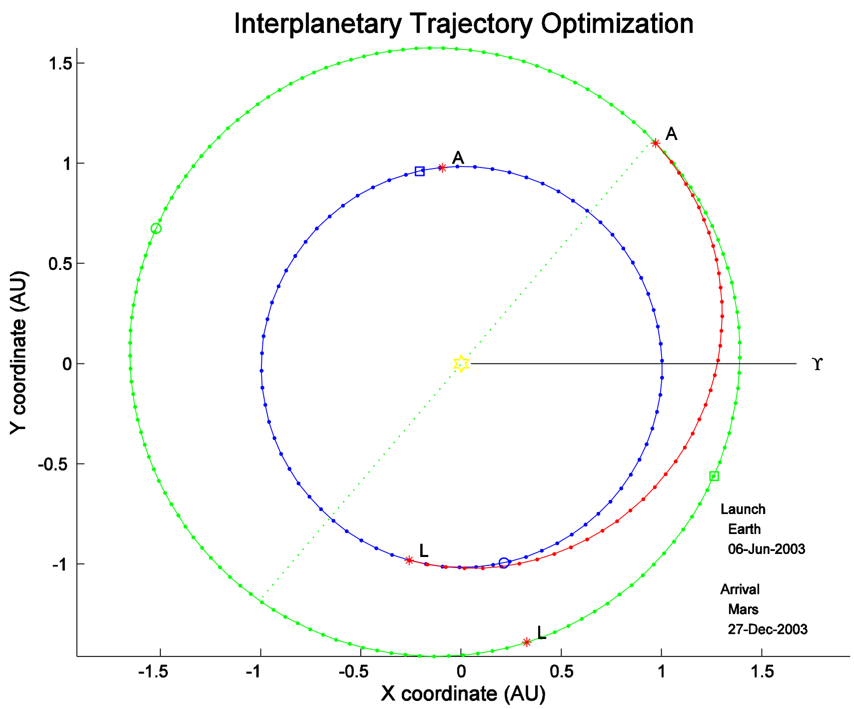
```
please input the plot step size (days)
?
```

A plot step size between 5 and 10 days is recommended.

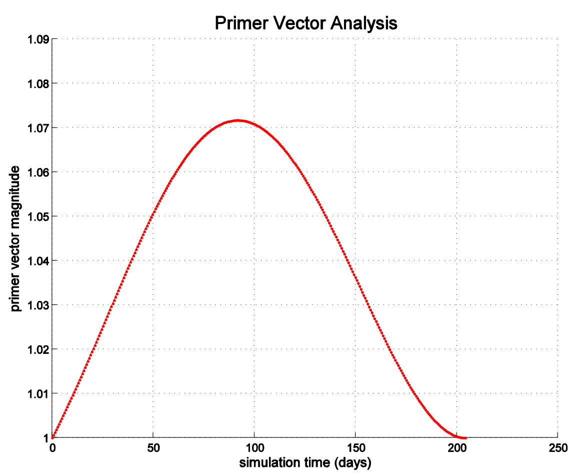
The `ipto_matlab` script will also create a color Postscript graphics file with TIFF preview of the trajectory using the following line of MATLAB source code.

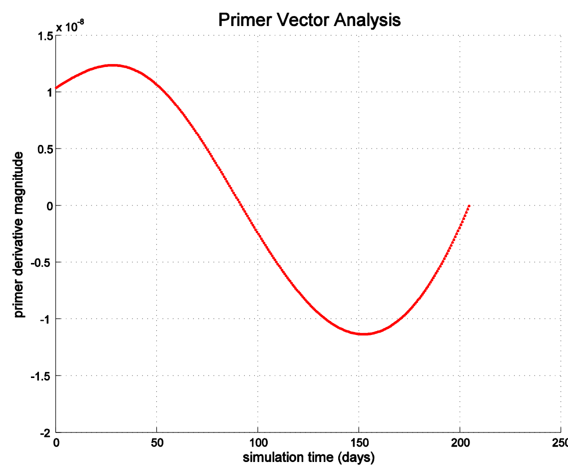
```
print -depsc -tiff -r300 ipto_matlab.eps
```

The following is a typical graphics display created with this MATLAB script. The plot is a *north ecliptic* view where we are looking down on the ecliptic plane from the north celestial pole. The vernal equinox direction is the labeled line pointing to the right, the launch body is labeled with an L and the arrival body is labeled with an A. The location of the launch and arrival celestial bodies at the launch time is marked with an asterisk. The initial orbit trace is blue, the transfer trajectory is red and the final orbit trace is green. The small squares are the perihelion location and the small circle is aphelion of the planetary orbits. If Earth is the departure planet, a green dotted line will be included and represents the line of nodes of the two planetary orbits.



The following are graphic displays of the magnitudes of the primer vector and its derivative for this example. From these two plots we can see the solution found by the `ipto_matlab` software is not two impulse optimal according to primer vector theory summarized in the Primer Vector Analysis section which begins on page 14 of this document.





Technical Discussion

An initial guess for the launch and arrival impulsive delta-v vectors can be determined from the solution of the Lambert two-point boundary-value problem (TPBVP). Lambert's Theorem states that the time to traverse a trajectory depends only upon the length of the semimajor axis a of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to a central body, and the length c of the chord joining these two positions.

The Lambert solution that initializes the `ipto_matlab` software uses the user's initial guess for the launch and arrival calendar dates.

Lambert's Problem

Lambert's problem is concerned with the determination of an orbit that passes between two positions within a specified time-of-flight. This classic astrodynamic problem is also known as the orbital two-point boundary value problem (TPBVP).

The time to traverse a trajectory depends only upon the length of the semimajor axis *a* of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to a central body, and the length *c* of the chord joining these two positions. This relationship can be stated as follows:

$$tof = tof\left(r_i + r_f, c, a\right)$$

From the following form of Kepler's equation

$$t - t_0 = \sqrt{\frac{a^3}{\mu}\left(E - e\sin E\right)}$$

we can write

$$t = \sqrt{\frac{a^3}{\mu}}\left[E - E_0 - e\left(\sin E - \sin E_0\right)\right]$$

where *E* is the eccentric anomaly associated with radius *r*, E_0 is the eccentric anomaly at r_0, and $t = 0$ when $r = r_0$.

At this point we need to introduce the following trigonometric sun and difference identities:

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{a+\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{a+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\cos\frac{a+\beta}{2}$$

If we let $E = \alpha$ and $E_0 = \beta$ and substitute the first trig identity into the second equation above, we have the following equation:

$$t = \sqrt{\frac{a^3}{\mu}}\left\{E - E_0 - 2\sin\frac{E-E_0}{2}\left(e\cos\frac{E+E_0}{2}\right)\right\}$$

With the two substitutions given by

$$e\cos\frac{E+E_0}{2}=\cos\frac{\alpha+\beta}{2} \qquad \sin\frac{E-E_0}{2}=\sin\frac{\alpha-\beta}{2}$$

the time equation becomes

$$t=\sqrt{\frac{a^3}{\mu}\left\{(\alpha-\beta)-2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}\right\}}$$

From the elliptic relationships given by

$$r=a(1-e\cos E)$$

$$x=a(\cos E-e)$$

$$y=a\sin E\sqrt{1-e^2}$$

and some more manipulation, we have the following equations:

$$\cos\alpha=\left(1-\frac{r+r_0}{2a}\right)-\frac{c}{2a}=1-\frac{r+r_0+c}{2a}=1-\frac{s}{a}$$

$$\sin\beta=\left(1-\frac{r+r_0}{2a}\right)+\frac{c}{2a}=1-\frac{r+r_0-c}{2a}=1-\frac{s-c}{a}$$

This part of the derivation makes use of the following three relationships:

$$\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}=1-\frac{r+r_0}{2}$$

$$\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}=\sin\frac{E-E_0}{2}\sqrt{1-\left(e\cos\frac{E+E_0}{2}\right)^2}$$

$$\left(\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}\right)^2=\left(\frac{x-x_0}{2a}\right)^2+\left(\frac{y-y_0}{2a}\right)^2=\left(\frac{c}{2a}\right)^2$$

With the use of the half angle formulas given by

$$\sin\frac{\alpha}{2}=\sqrt{\frac{s}{2a}} \qquad \sin\frac{\beta}{2}=\sqrt{\frac{s-c}{2a}}$$

and several additional substitutions, we have the time-of-flight form of Lambert's theorem

$$t=\sqrt{\frac{a^3}{\mu}}\left[(\alpha-\beta)-(\sin\alpha-\sin\beta)\right]$$

A discussion about the angles α and β can be found in "Geometrical Interpretation of the Angles α and β in Lambert's Problem" by J. E. Prussing, AIAA *Journal of Guidance and Control*, Volume 2, Number 5, Sept.-Oct. 1979, pages 442-443.

The algorithm used in this MATLAB script is based on the method described in "A Procedure for the Solution of Lambert's Orbital Boundary-Value Problem" by R. H. Gooding, *Celestial Mechanics and Dynamical Astronomy* **48:** 145-165, 1990. This iterative solution is valid for elliptic, parabolic and hyperbolic transfer orbits which may be either posigrade or retrograde, and involve one or more revolutions about the central body.

The planetary ephemeris implemented in this MATLAB script is based on the JPL DE424 ephemeris.

Important Note

The binary ephemeris file provided with this computer program was created for using a Windows compatible computer. For other platforms, you will need to create or obtain binary files specific to that system. Information and computer programs for creating these files can be found at the JPL solar system FTP site located at ftp://ssd.jpl.nasa.gov/pub/. This site provides ASCII data files and Fortran computer programs for creating a binary file. A program for testing the user's ephemeris is also provided along with documentation.

In order to model ballistic interplanetary missions involving asteroids and comets, the classical orbital elements of an asteroid or comet relative to the mean ecliptic and equinox of J2000 coordinate system must be provided by the user. These elements can be obtained from the JPL Near Earth Object (NEO) website (http://neo.jpl.nasa.gov).

These orbital elements consist of the following items:

- calendar date of perihelion passage
- perihelion distance (AU)
- orbital eccentricity (non-dimensional)
- orbital inclination (degrees)
- argument of perihelion (degrees)
- longitude of ascending node (degrees)

The software determines the mean anomaly of the asteroid or comet at any simulation time using the following equation:

$$M = \sqrt{\frac{\mu_s}{a^3}}\, t_{pp} = \sqrt{\frac{\mu_s}{a^3}}\left(JD - JD_{pp}\right)$$

where μ_s is the gravitational constant of the sun, a is the semimajor axis of the celestial body, and t_{pp} is the time since perihelion passage.

The semimajor axis is determined from the perihelion distance r_p and orbital eccentricity e according to

$$a = \frac{r_p}{(1-e)}$$

This solution of Kepler's equation in this MATLAB script is based on a numerical solution devised by Professor J.M.A. Danby at North Carolina State University. Additional information about this algorithm can be found in "The Solution of Kepler's Equation", *Celestial Mechanics*, **31** (1983) 95-107, 317-328 and **40** (1987) 303-312.

The initial guess for Danby's method is

$$E_0 = M + 0.85\,\mathrm{sign}(\sin M)\,e$$

The fundamental equation we want to solve is

$$f(E) = E - e\sin E - M = 0$$

which has the first three derivatives given by

$$f'(E) = 1 - e\cos E \qquad f''(E) = e\sin E \qquad f'''(E) = e\cos E$$

The iteration for an updated eccentric anomaly based on a current value E_n is given by the next four equations:

$$\Delta(E_n) = -\frac{f}{f'} \qquad \Delta^*(E_n) = -\frac{f}{f' + \frac{1}{2}\Delta f''}$$

$$\Delta_n(E_n) = -\frac{f}{f' + \frac{1}{2}\Delta f'' + \frac{1}{6}\Delta^{*2} f'''} \qquad E_{n+1} = E_n + \Delta_n$$

This algorithm provides quartic convergence of Kepler's equation. This process is repeated until the following convergence test involving the fundamental equation is satisfied:

$$\left| f(E) \right| \le \varepsilon$$

where ε is the convergence tolerance. This tolerance is hardwired in the software to ε =1.0e-10. Finally, the true anomaly can be calculated with the following two equations

$$\sin\theta = \sqrt{1-e^2}\,\sin E \qquad \cos\theta = \cos E - e$$

and the four quadrant inverse tangent given by

$$\theta = \tan^{-1}(\sin\theta, \cos\theta)$$

If the orbit is hyperbolic, the initial guess is

$$H_0 = \log\left(\frac{2M}{e} + 1.8\right)$$

where H_0 is the hyperbolic anomaly. The fundamental equation and first three derivatives for this case are as follows:

$$f(H) = e\sinh H - H - M \qquad f'(H) = e\cosh H - 1$$

$$f''(H) = e\sinh H \qquad f'''(H) = e\cosh H$$

Otherwise, the iteration loop which calculates Δ, Δ^*, and so forth is the same. The true anomaly for hyperbolic orbits is determined with this next set of equations:

$$\sin\theta = \sqrt{e^2 - 1}\sinh H \qquad \cos\theta = e - \cosh H$$

The true anomaly is then determined from a four quadrant inverse tangent evaluation of these two equations.

The $\Delta V's$ required at launch and arrival are simply the differences between the velocity on the transfer trajectory determined by the solution of Lambert's problem and the heliocentric velocities of the two celestial bodies. If we treat each celestial body as a point mass and assume *impulsive* maneuvers, the *body-centered* magnitude and direction of the required maneuvers are given by the two vector equations:

$$\Delta\mathbf{V}_L = \mathbf{V}_{T_L} - \mathbf{V}_{B_L}$$

$$\Delta\mathbf{V}_A = \mathbf{V}_{B_A} - \mathbf{V}_{T_A}$$

where

$\mathbf{V}_{T_L}$ = heliocentric velocity vector of the transfer trajectory at launch

$\mathbf{V}_{T_A}$ = heliocentric velocity vector of the transfer trajectory at arrival

$\mathbf{V}_{B_L}$ = heliocentric velocity vector of the celestial body at launch

$\mathbf{V}_{B_A}$ = heliocentric velocity vector of the celestial body at arrival

The scalar magnitude of each maneuver is also called the "hyperbolic excess velocity" or v-infinity at launch and arrival. The hyperbolic excess velocity is the speed of the spacecraft relative to each celestial body at an *infinite* distance from the body. Furthermore, the *energy* or C_3 at launch or arrival is equal to v_∞^2 for the respective maneuver. C_3 is also equal to <u>twice</u> the orbital energy per unit mass (the specific orbital energy).

The orientation of the departure and arrival hyperbolas is specified in terms of the right ascension and declination of the asymptote. These coordinates can be calculated using the components of the V_∞ velocity vector.

The right ascension of the asymptote is determined from $\alpha = \tan^{-1}\left(\Delta V_y, \Delta V_z\right)$ and the geocentric declination of the asymptote is given by $\delta = 90^0 - \cos^{-1}\left(\Delta \hat{V}_z\right)$ where $\Delta \hat{V}_z$ is z-component of the unit ΔV vector.

In this MATLAB script the heliocentric planetary coordinates are computed in the J2000 mean equator and equinox coordinate system (EME2000). These coordinates are transformed to the mean ecliptic and equinox of J2000 with the following matrix-vector operation.

$$\mathbf{S}_{ec} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{bmatrix} \mathbf{S}_{eq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.917482062069182 & 0.397777155931914 \\ 0 & -0.397777155931914 & 0.917482062069182 \end{bmatrix} \mathbf{S}_{eq}$$

where $\mathbf{S}_{ec}$ is the state vector (position and velocity vectors) in the ecliptic frame, $\mathbf{S}_{eq}$ is the state vector in the Earth equatorial frame and ε is the mean obliquity of the ecliptic at J2000. The J2000 value of the mean obliquity of the ecliptic is equal to $\varepsilon = 23^{\circ}26'21.''448$.

The following figure illustrates the geometry of the EME2000 coordinate system. The origin of this heliocentric inertial coordinate system is the Sun and the fundamental plane is the Earth's mean equator. The z-axis of this system is normal to the Earth's mean equator at epoch J2000, the x-axis is parallel to the vernal equinox of the Earth's mean orbit at epoch J2000, and the y-axis completes the right-handed coordinate system. The epoch J2000 is the Julian Date 2451545.0 which corresponds to January 1, 2000, 12 hours Terrestrial Time (TT).

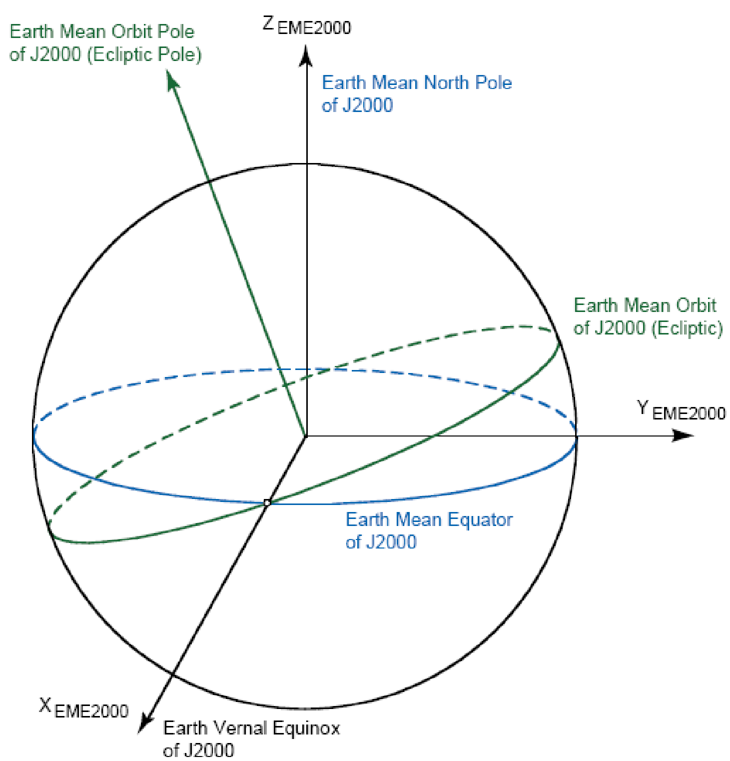


Figure 1. Earth mean equator and equinox of J2000 coordinate system

Trajectory Optimization

In the terminology of numerical optimization, this MATLAB script treats the launch and arrival dates as *control or optimization variables* and attempts to minimize the launch, arrival or sum of launch and arrival scalar $\Delta V's$. The scalar magnitude of the selected ΔV is called *the objective function* or *the performance index.*

The software can solve for the following types of optimized interplanetary space missions:

- minimum launch ΔV
- minimum arrival ΔV
- minimum total ΔV

As noted in "On the Nature of Earth-Mars Porkchop Plots", the optimality of two impulse interplanetary trajectories is governed by the following three factors

1) Departure and arrival near the line of nodes
2) Arrival at the aphelion of the more eccentric celestial body orbit
3) Departure from the perihelion of the less eccentric celestial body orbit

where item 1 is the most important and item 3 the least important.

Primer Vector Analysis

This section summarizes the primer vector analysis performed by `ipto_matlab` software. The term primer vector was invented by Derek F. Lawden and represents the adjoint vector for velocity. A technical discussion about primer theory can be found in Lawden's classic text, *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963. Another excellent resource is "Primer Vector Theory and Applications", Donald J. Jezewski, NASA TR R-454, November 1975, along with "Optimal, Multi-burn, Space Trajectories", also by Jezewski.

As shown by Lawden, the following four necessary conditions must be satisfied in order for an impulsive orbital transfer to be *locally optimal*:

(1) the primer vector and its first derivative are everywhere continuous

(2) whenever a velocity impulse occurs, the primer is a unit vector aligned with the impulse and has unit magnitude $\left(\mathbf{p} = \hat{\mathbf{p}} = \hat{\mathbf{u}}_T \text{ and } \|\mathbf{p}\| = 1\right)$

(3) the magnitude of the primer vector may not exceed unity on a coasting arc $\left(\|\mathbf{p}\| = p \le 1\right)$

(4) at all interior impulses (not at the initial or final times) $\mathbf{p} \bullet \dot{\mathbf{p}} = 0$; therefore, $d\|\mathbf{p}\|/dt = 0$ at the intermediate impulses

Furthermore, the scalar magnitude of the primer vector derivative at the initial and final impulses provide information about how to improve the nominal transfer trajectory by changing the endpoint times and/or moving the velocity impulse times. These four cases for non-zero slopes are summarized as follows;

- If $\dot{p}_0 > 0$ and $\dot{p}_f < 0 \rightarrow$ perform an initial coast before the first impulse and add a final coast after the second impulse
- If $\dot{p}_0 > 0$ and $\dot{p}_f > 0 \rightarrow$ perform an initial coast before the first impulse and move the second impulse to a later time
- If $\dot{p}_0 < 0$ and $\dot{p}_f < 0 \rightarrow$ perform the first impulse at an earlier time and add a final coast after the second impulse
- If $\dot{p}_0 < 0$ and $\dot{p}_f > 0 \rightarrow$ perform the first impulse at an earlier time and move the second impulse to a later time

The primer vector analysis of a two impulse orbital transfer involves the following steps.

First partition the two-body state transition matrix $\Phi(t,t_0)$ as follows:

$$\Phi(t,t_0) = \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{r}}{\partial \mathbf{v}_0} \\ \dfrac{\partial \mathbf{v}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{v}}{\partial \mathbf{v}_0} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}$$

where

$$\Phi_{11} = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \right] = \begin{bmatrix} \partial x / \partial x_0 & \partial x / \partial y_0 & \partial x / \partial z_0 \\ \partial y / \partial x_0 & \partial y / \partial y_0 & \partial y / \partial z_0 \\ \partial z / \partial x_0 & \partial z / \partial y_0 & \partial z / \partial z_0 \end{bmatrix}$$

and so forth.

The value of the primer vector at any time *t* along a two body trajectory is given by

$$\mathbf{p}(t) = \Phi_{11}(t,t_0)\mathbf{p}_0 + \Phi_{12}(t,t_0)\dot{\mathbf{p}}_0$$

and the value of the primer vector derivative is

$$\dot{\mathbf{p}}(t) = \Phi_{21}(t,t_0)\mathbf{p}_0 + \Phi_{22}(t,t_0)\dot{\mathbf{p}}_0$$

which can also be expressed as

$$\begin{Bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{Bmatrix} = \Phi(t,t_0) \begin{Bmatrix} \mathbf{p}_0 \\ \dot{\mathbf{p}}_0 \end{Bmatrix}$$

The primer vector boundary conditions at the initial and final impulses are as follows:

$$\mathbf{p}(t_0) = \mathbf{p}_0 = \frac{\Delta \mathbf{V}_0}{|\Delta \mathbf{V}_0|} \qquad \mathbf{p}(t_f) = \mathbf{p}_f = \frac{\Delta \mathbf{V}_f}{|\Delta \mathbf{V}_f|}$$

These two conditions illustrate that at the locations of velocity impulses, the primer vector is a unit vector in the direction of the velocity impulses.

The value of the primer vector derivative at the initial time is

$$\dot{\mathbf{p}}(t_0) = \dot{\mathbf{p}}_0 = \Phi_{12}^{-1}(t_f, t_0)\left\{\mathbf{p}_f - \Phi_{11}(t_f, t_0)\mathbf{p}_0\right\}$$

provided the Φ_{12} sub-matrix is non-singular.

The scalar magnitude of the derivative of the primer vector can be determined from

$$\frac{d\|\mathbf{p}\|}{dt} = \frac{d}{dt}(\mathbf{p}\bullet\mathbf{p})^2 = \frac{\dot{\mathbf{p}}\bullet\mathbf{p}}{\|\mathbf{p}\|}$$

SNOPT algorithm implementation

This section provides details about the part of the `ipto_matlab` MATLAB script that solves this nonlinear programming (NLP) problem using the SNOPT algorithm. In this classic trajectory optimization problem, the launch and arrival calendar dates are the *control variables* and the scalar ΔV computed by the solution of Lambert's problem is the *objective function* or *performance index*.

MATLAB versions of SNOPT for several computer platforms can be found at Professor Philip Gill's web site which is located at http://scicomp.ucsd.edu/~peg/.

The SNOPT algorithm requires an initial guess for the control variables. For this problem they are computed with the following code

```
xg(1) = jdate1 - jdate0;

xg(2) = jdate2 - jdate0;

xg = xg';
```

where `jdate1` and `jdate2` are the initial user-provided launch and arrival date guesses, and `jdate0` is a reference Julian date equal to `2451544.5` (January 1, 2000). This offset value is used to *scale* the Julian Date control variables.

The algorithm also requires lower and upper bounds for the control variables. These are determined from the initial guesses and user-defined search boundaries as follows:

```
% bounds on control variables

xlwr(1) = xg(1) - ddays1;
xupr(1) = xg(1) + ddays1;
```

```
xlwr(2) = xg(2) - ddays2;
xupr(2) = xg(2) + ddays2;

xlwr = xlwr';
xupr = xupr';

xlwr = xlwr';
xupr = xupr';
```

where ddays1 and ddays2 are the user-defined launch and arrival search boundaries, respectively.

If the mission constraints option is in effect, the following code defines the lower and upper bounds for each of the four nonlinear mission constraints.

```
if (imcon == 1)

   % bounds on nonlinear constraints

   flow(2) = flow_c3;
   fupp(2) = fupp_c3;

   flow(3) = dtr * flow_dla;
   fupp(3) = dtr * fupp_dla;

   flow(4) = flow_tof;
   fupp(4) = fupp_tof;

   flow(5) = flow_vinf;
   fupp(5) = fupp_vinf;

end
```

The algorithm also requires lower and upper bounds on the objective function. For this problem these bounds are given by

```
% bounds on objective function

flow(1) = 0.0d0;
fupp(1) = +Inf;
```

The actual call to the SNOPT MATLAB interface function is as follows

```
[x, f, inform, xmul, fmul] = snopt(xg, xlwr, xupr, flow, fupp, 'iptofunc');
```

where iptofunc is the name of the MATLAB function that solves Lambert's problem and computes the current value of the objective function and any required nonlinear constraints.

The ipto_matlab script will also read an SNOPT SPECS file. For this example the contents of this file are as follows:

```
Begin SNOPT options
    minor iterations limit        1000
    derivative option             0
    major optimality tolerance    1.0d-6
    solution                      Yes
End SNOPT options
```

Please consult the SNOPT documentation for a complete explanation of the SPECS file. A PDF version of the SNOPT user's manual is also available at Professor Gill's website.

Algorithm Resources

Modern Astrodynamics, Victor R. Bond and Mark C. Allman, Princeton University Press, 1996.

"The Solution of Kepler's Equation", J. M. A. Danby, *Celestial Mechanics*, **31** (1983) 95-107, 317-328 and **40** (1987) 303-312.

"A Procedure for the Solution of Lambert's Orbital Boundary-Value Problem", R. H. Gooding, *Celestial Mechanics and Dynamical Astronomy* **48:** 145-165, 1990.

"The Planetary and Lunar Ephemeris DE 421", W. M. Folkner, J. G. Williams, and D. H. Boggs, JPL IOM 343R-08-003, 31 March 2008.

"Report of the IAU/IAG Working Group on Cartographic Coordinates and Rotational Elements of the Planets and Satellites: 2000", *Celestial Mechanics and Dynamical Astronomy*, **82**: 83-110, 2002.

"IERS Conventions (2003)", IERS Technical Note 32, November 2003.

"Planetary Constants and Models", R. Vaughan, JPL D-12947, December 1995.

A. B. Sergeyevsky, G. C. Synder, and R. A. Cunniff, "Interplanetary Mission Design Handbook, Volume 1, Part 2", JPL Publication 82-43, September 15, 1983.

"Rapid Solar Sail Rendezvous Missions to Asteroid 99942 Apophis", Giovanni Mengali and Alessandro A. Quarta, AIAA *Journal of Spacecraft and Rockets*, Vol. 46, No. 1, January-February 2009, pp. 134-140.

"Optimal Interplanetary Orbit Transfers by Direct Transcription", John T. Betts, *The Journal of the Astronautical Sciences*, Vol. 42, No. 3, July-September 1994, pp. 247-268.

Ryan C. Woolley and Charles W. Whetsel, "On the Nature of Earth-Mars Porkchop Plots", AAS 13-226, 23rd AAS/AIAA Space Flight Mechanics Meeting, Kauai, Hawaii, February 10-14, 2013.

S. Matousek and A. B. Sergeyevsky, "To Mars and Back: 2002-2020 Ballistic Trajectory Data for the Mission Architect", AIAA 98-4396, AIAA/AAS Astrodynamics Specialist Conference, Boston, MA, August 10-12, 1998.

L. E. George and L. D. Koss, "Interplanetary Mission Design Handbook: Earth-to-Mars Mission Opportunities and Mars-to-Earth Return Opportunities 2009-2024", NASA TM-1998-208533, 1998.

APPENDIX A

Earth-to-Tempel 1 Trajectory Analysis

This appendix summarizes typical trajectory characteristics of a ballistic and patched-conic mission from Earth to the comet Tempel 1. This simulation example minimizes the magnitude of the departure delta-v at the Earth departure.

Here's the `ipto_matlab` user interaction and the script output for this example.

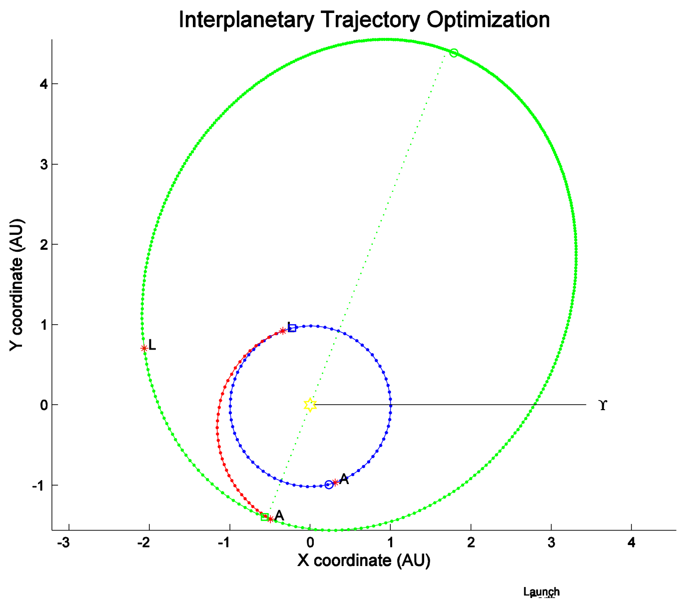
```
              program ipto_matlab

   < interplanetary trajectory optimization >


   departure conditions - start date

   please input the calendar date
   (1 <= month <= 12, 1 <= day <= 31, year = all digits!)
   ? 12,1,2004

   please input the departure date search boundary in days
   ? 60


   arrival conditions - start date

   please input the calendar date
   (1 <= month <= 12, 1 <= day <= 31, year = all digits!)
   ? 7,1,2005

   please input the arrival date search boundary in days
   ? 90

    celestial body menu

     <1>  Mercury
     <2>  Venus
     <3>  Earth
     <4>  Mars
     <5>  Jupiter
     <6>  Saturn
     <7>  Uranus
     <8>  Neptune
     <9>  Pluto
     <10> asteroid/comet

   please select the departure celestial body
   ? 3

    celestial body menu

     <1>  Mercury
     <2>  Venus
     <3>  Earth
     <4>  Mars
     <5>  Jupiter
     <6>  Saturn
     <7>  Uranus
     <8>  Neptune
     <9>  Pluto
     <10> asteroid/comet
```

```
please select the arrival celestial body
? 10


would you like to enforce mission constraints (y = yes, n = no)
? n

      optimization menu

 <1> minimize departure delta-v

 <2> minimize arrival delta-v

 <3> minimize total delta-v

 <4> no optimization

 selection (1, 2, 3 or 4)
? 1


       program ipto_matlab

   minimize departure delta-v

departure celestial body     Earth

departure calendar date      10-Jan-2005
departure TDB time           08:46:54.744

departure julian date        2453380.8659


arrival celestial body       asteroid/comet

arrival calendar date        10-Jul-2005
arrival TDB time             02:24:29.401

arrival julian date          2453561.6003

transfer time                  180.7344  days

heliocentric orbital conditions prior to the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.4974235642e+08  1.7610519542e-02  7.7741153172e-04  3.1647547001e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 1.4672891357e+02  6.9458397394e+00  3.2342130975e+02  3.6578618874e+02

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -5.06816788009146e+07  +1.38118931783621e+08  -1.18960940212756e+03  +1.47124001728279e+08

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 -2.84629803229640e+01  -1.03767268372939e+01  +3.29584858730136e-04  +3.02955064131610e+01


heliocentric orbital conditions after the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.9458780387e+08  2.4392147259e-01  5.7287208652e-01  1.8032512802e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 2.9010388993e+02  3.5972120768e+02  1.8004633570e+02  5.4185580915e+02

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -5.06816788009146e+07  +1.38118931783621e+08  -1.18960940212756e+03  +1.47124001728279e+08
```

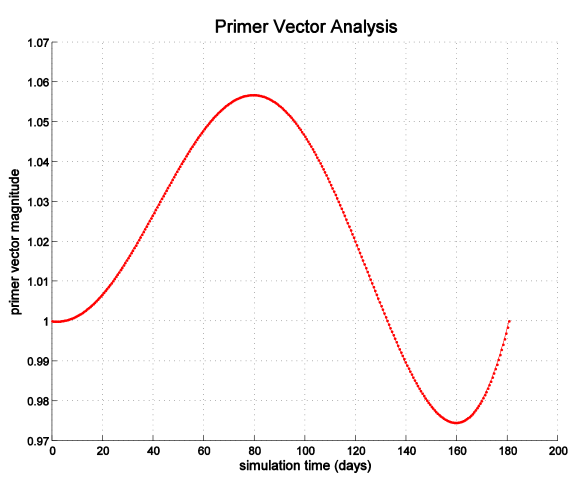
```
        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 -3.14344527003716e+01  -1.15686784469718e+01  -3.34917119013286e-01  +3.34973328349876e+01


heliocentric orbital conditions prior to the second maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.9458780387e+08  2.4392147259e-01  5.7287208652e-01  1.8032512802e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 2.9010388993e+02  1.4049056085e+02  3.2081568887e+02  5.4185580915e+02

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -7.36868621538256e+07  -2.13047243958176e+08  -1.42409963263081e+06  +2.25434934897001e+08

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 +1.92933379365169e+01  -1.10958802607591e+01  +1.43022352008143e-01  +2.22569517878554e+01


heliocentric orbital conditions after the second maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 4.6697445253e+08  5.1749100000e-01  1.0530100000e+01  1.7883900000e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 6.8973400000e+01  3.1419160077e+00  1.8198091601e+02  2.0144198451e+03

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -7.36868621538256e+07  -2.13047243958176e+08  -1.42409963263081e+06  +2.25434934897001e+08

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 +2.75933039187317e+01  -1.00985027120853e+01  -5.46110314910779e+00  +2.98863484852647e+01


heliocentric orbital conditions of arrival body
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 4.6697445253e+08  5.1749100000e-01  1.0530100000e+01  1.7883900000e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 6.8973400000e+01  3.1419160077e+00  1.8198091601e+02  2.0144198451e+03

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -7.36868621538243e+07  -2.13047243958177e+08  -1.42409963263081e+06  +2.25434934897001e+08

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 +2.75933039187318e+01  -1.00985027120852e+01  -5.46110314910779e+00  +2.98863484852647e+01


departure delta-v and energy requirements
(mean equator and equinox of J2000)
-----------------------------------

x-component of delta-v      -2971.472377  meters/second
y-component of delta-v       -960.240740  meters/second
z-component of delta-v       -781.713958  meters/second

delta-v magnitude            3219.128311  meters/second

energy                         10.362787  kilometers^2/second^2

asymptote right ascension     197.908404  degrees
asymptote declination         -14.053869  degrees


arrival delta-v and energy requirements
(mean equator and equinox of J2000)
-----------------------------------
```

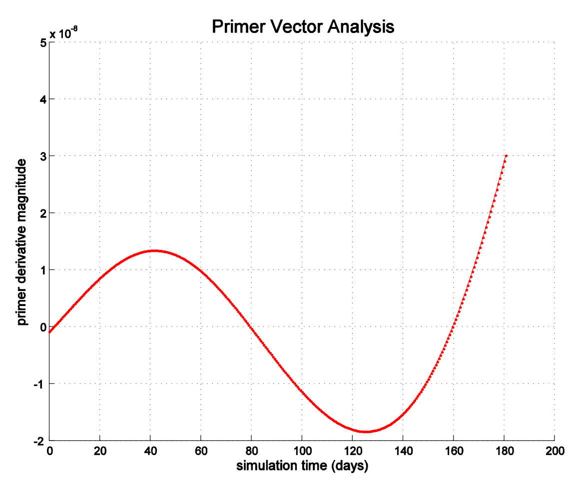
```
x-component of delta-v       -8299.965982  meters/second
y-component of delta-v       -3144.269113  meters/second
z-component of delta-v        4744.950616  meters/second

delta-v magnitude            10064.314180  meters/second

energy                         101.290420  kilometers^2/second^2

asymptote right ascension      200.748149  degrees
asymptote declination           28.129298  degrees

total delta-v                13283.442491  meters/second

total energy                   111.653207  kilometers^2/second^2
```

Here's the graphics display of the interplanetary transfer trajectory along with the heliocentric orbits of the Earth and Tempel 1. We can see from this graphics display and the screen display data above (heliocentric true anomaly $\approx$ 3 degrees) that spacecraft encounter with Tempel 1 occurs near perihelion of the comet's orbit.



The following are graphic displays of the magnitudes of the primer vector and its derivative for this orbit transfer example. From these two plots we can see that the solution found by the `ipto_matlab` script is not optimal according to primer vector theory.





APPENDIX B

Mission Constraints Example

This appendix provides a trajectory example which demonstrates the mission constraint option of the `ipto_matlab` MATLAB script. When exercising the mission constraints program option, you may receive the following screen display if all the mission constraints have not been satisfied.

```
check solution!!

all mission constraints may not be satisfied
```

If you receive this message, consider "relaxing" the mission constraint or constraints that are not satisfied. It may also be helpful to run your example with the mission constraint graphics option of the `porkchop_ftn` computer program and examine the solution geometry.

The launch delta-v of the Earth-to-Mars mission opportunity in 2011 is optimized with the following user-defined mission constraints;

$$6.0 \le C_{3_L} \le 10.0$$

$$-28.5^\circ \le DLA \le +28.5^\circ$$

$$100.0 \le TOF \le 300.0$$

$$1.0 \le V_{\infty_a} \le 3.0$$

where C_{3_L} is the departure or launch energy in kilometer per second quantity squared, DLA is the geocentric EME2000 declination of the launch hyperbola in degrees, TOF is the time-of-flight in days, and V_{∞_a} is the scalar magnitude of the arrival v-infinity vector at Mars in kilometers per second.

The following is the `ipto_matlab` user interaction for this example.

```
            program ipto_matlab

< interplanetary trajectory optimization >


departure conditions - start date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 11,17,2011

please input the departure date search boundary in days
? 60


arrival conditions - start date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 8,11,2012

please input the arrival date search boundary in days
? 60
```

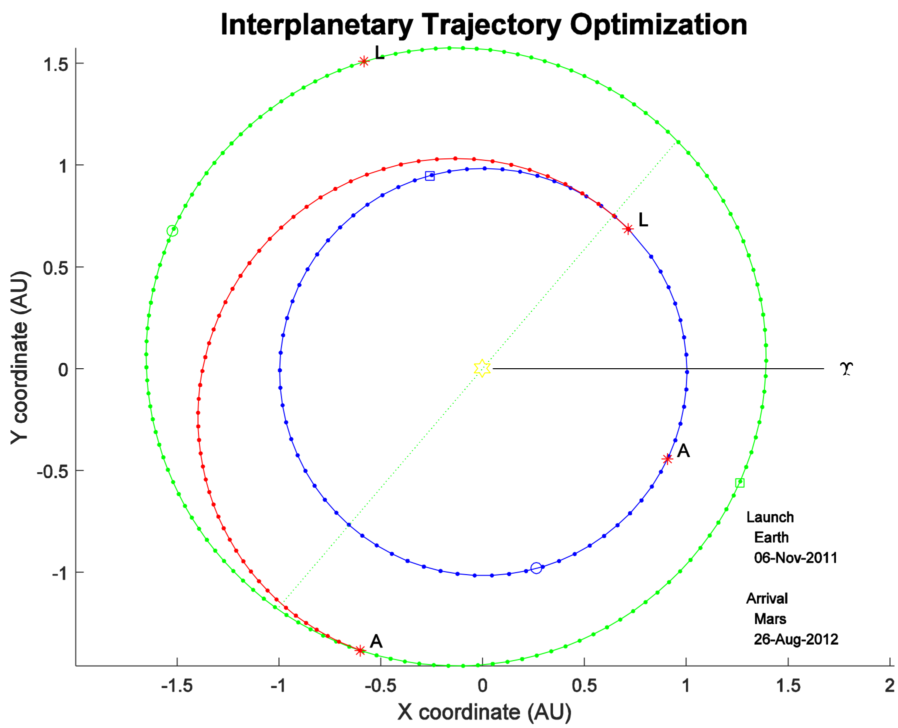
```
celestial body menu

  <1>  Mercury
  <2>  Venus
  <3>  Earth
  <4>  Mars
  <5>  Jupiter
  <6>  Saturn
  <7>  Uranus
  <8>  Neptune
  <9>  Pluto
  <10> asteroid/comet

please select the departure celestial body
? 3

celestial body menu

  <1>  Mercury
  <2>  Venus
  <3>  Earth
  <4>  Mars
  <5>  Jupiter
  <6>  Saturn
  <7>  Uranus
  <8>  Neptune
  <9>  Pluto
  <10> asteroid/comet

please select the arrival celestial body
? 4


would you like to enforce mission constraints (y = yes, n = no)
? y

please input the lower bound for departure C3 (km^2/sec^2)
? 6

please input the upper bound for departure C3 (km^2/sec^2)
? 10

please input the lower bound for departure DLA (degrees)
? -28.5

please input the upper bound for departure DLA (degrees)
? 28.5

please input the lower bound for time-of-flight (days)
? 100

please input the upper bound for time-of-flight (days)
? 300

please input the lower bound for arrival v-infinity (km/sec)
? 1

please input the upper bound for arrival v-infinity (km/sec)
? 3

      optimization menu

 <1> minimize departure delta-v

 <2> minimize arrival delta-v

 <3> minimize total delta-v

 <4> no optimization

 selection (1, 2, 3 or 4)
? 1
```

The following is the screen output created by `ipto_matlab` for this example.

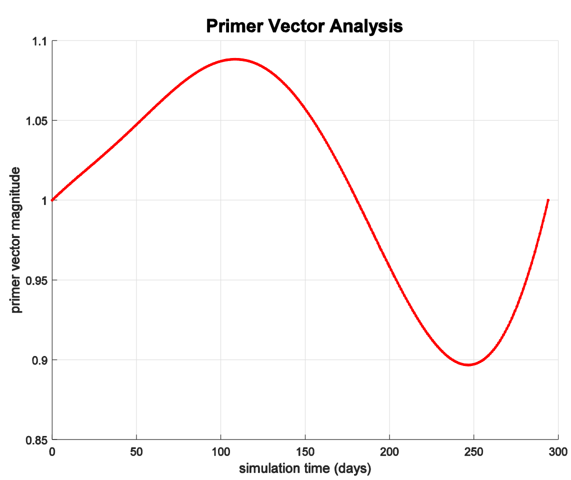
```
       program ipto_matlab

   minimize departure delta-v

departure celestial body     Earth

departure calendar date      06-Nov-2011
departure TDB time           19:58:30.582

departure julian date        2455872.3323


arrival celestial body       Mars

arrival calendar date        26-Aug-2012
arrival TDB time             19:20:07.434

arrival julian date          2456166.3056

transfer time                  293.9733  days

heliocentric orbital conditions prior to the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.4950950904e+08  1.6641163552e-02  1.5240318293e-03  3.0847852177e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 1.5664975134e+02  2.9876217025e+02  2.4724069202e+02  3.6493333148e+02

        rx (km)                ry (km)               rz (km)               rmag (km)
 +1.06861025481163e+08  +1.02800338166280e+08  -3.63707482737303e+03  +1.48280775244856e+08

        vx (kps)               vy (kps)              vz (kps)              vmag (kps)
 -2.11368237379601e+01  +2.13447410995149e+01  -2.98418119946575e-04  +3.00393623504889e+01


heliocentric orbital conditions after the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.8773721478e+08  2.1185066927e-01  1.4070724552e+00  8.8956563671e+00

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.3947658224e+01  3.5104711150e+02  3.5994276787e+02  5.1349458076e+02

        rx (km)                ry (km)               rz (km)               rmag (km)
 +1.06861025481163e+08  +1.02800338166280e+08  -3.63707482737303e+03  +1.48280775244856e+08

        vx (kps)               vy (kps)              vz (kps)              vmag (kps)
 -2.34473547543591e+01  +2.30798404031567e+01  +8.07861196073348e-01  +3.29106383670094e+01


heliocentric orbital conditions prior to the second maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.8773721478e+08  2.1185066927e-01  1.4070724552e+00  8.8956563671e+00

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.3947658224e+01  1.9364416153e+02  2.0253981790e+02  5.1349458076e+02

        rx (km)                ry (km)               rz (km)               rmag (km)
 -9.00995880751167e+07  -2.07030644205069e+08  -2.12537376708756e+06  +2.25796679835912e+08

        vx (kps)               vy (kps)              vz (kps)              vmag (kps)
 +2.03489743966313e+01  -7.36818166666030e+00  -4.77187749954083e-01  +2.16471376448798e+01


heliocentric orbital conditions after the second maneuver
(mean ecliptic and equinox of J2000)
```

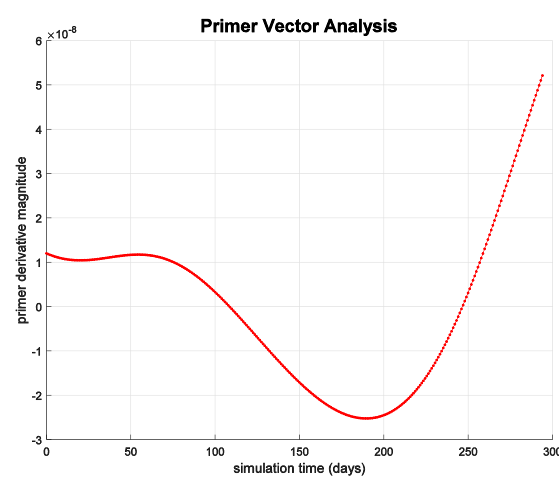
```
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2794532984e+08  9.3344391222e-02  1.8487297043e+00  2.8652221290e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9525660857e+01  2.7044180875e+02  1.9696402165e+02  6.8699939875e+02

        rx (km)                ry (km)               rz (km)               rmag (km)
 -9.00995880751167e+07  -2.07030644205069e+08  -2.12537376708756e+06  +2.25796679835912e+08

        vx (kps)               vy (kps)              vz (kps)              vmag (kps)
 +2.31325613200168e+01  -7.59264032750929e+00  -7.27063249673829e-01  +2.43575902202226e+01


heliocentric orbital conditions of arrival body
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2794532984e+08  9.3344391222e-02  1.8487297043e+00  2.8652221290e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9525660857e+01  2.7044180875e+02  1.9696402165e+02  6.8699939875e+02

        rx (km)                ry (km)               rz (km)               rmag (km)
 -9.00995880751277e+07  -2.07030644205065e+08  -2.12537376708730e+06  +2.25796679835913e+08

        vx (kps)               vy (kps)              vz (kps)              vmag (kps)
 +2.31325613200162e+01  -7.59264032751061e+00  -7.27063249673843e-01  +2.43575902202225e+01


departure delta-v and energy requirements
(mean equator and equinox of J2000)
-----------------------------------

x-component of delta-v      -2310.531016  meters/second
y-component of delta-v       1270.455054  meters/second
z-component of delta-v       1431.654816  meters/second

delta-v magnitude            3000.374166  meters/second

energy                          9.002245  kilometers^2/second^2

asymptote right ascension     151.195623  degrees
asymptote declination          28.500000  degrees


arrival delta-v and energy requirements
(mean equator and equinox of J2000)
-----------------------------------

x-component of delta-v      -2783.586923  meters/second
y-component of delta-v        106.542029  meters/second
z-component of delta-v        318.540816  meters/second

delta-v magnitude            2803.778810  meters/second

energy                          7.861176  kilometers^2/second^2

Mars-mean-equator and IAU node of epoch

asymptote right ascension     133.531925  degrees
asymptote declination         -21.579029  degrees


total delta-v                5804.152976  meters/second

total energy                   16.863421  kilometers^2/second^2
```

Here’s the graphics display of the interplanetary transfer trajectory along with the heliocentric orbits of the Earth and Mars.



The following are graphic displays of the magnitudes of the primer vector and its derivative for this orbit transfer example. From these two plots we can see that the solution found by the `ipto_matlab` script is not optimal according to primer vector theory.





APPENDIX C

Areocentric Coordinate Transformation

This appendix describes the transformation of coordinates between the Earth mean equator and equinox of J2000 (EME2000) and the areocentric mean equator and IAU node of epoch coordinate systems. This transformation is used to compute the right ascension and declination of the incoming asymptote at Mars arrival.

A unit vector in the direction of the pole of Mars can be determined from

$$\hat{\mathbf{p}}_{Mars} = \begin{bmatrix} \cos\alpha_p \cos\delta_p \\ \sin\alpha_p \cos\delta_p \\ \sin\delta_p \end{bmatrix}$$

The IAU 2000 right ascension and declination of the pole of Mars in the Earth mean equator and equinox of J2000 (EME2000) coordinate system are given by the following expressions

$$\alpha_p = 317.68143 - 0.1061T$$

$$\delta_p = 52.88650 - 0.0609T$$

where T is the time in Julian centuries given by $T = \left(JD - 2451545.0\right)/36525$ and JD is the TDB Julian Date.

The unit vector in the direction of the *IAU-defined* x-axis is computed from

$$\hat{\mathbf{x}} = \hat{\mathbf{p}}_{J2000} \times \hat{\mathbf{p}}_{Mars}$$

where $\hat{\mathbf{p}}_{J2000} = \left[0\;0\;1\right]^T$ is unit vector in the direction of the pole of the J2000 coordinate system.

The unit vector in the y-axis direction of this coordinate system is

$$\hat{\mathbf{y}} = \hat{\mathbf{p}}_{Mars} \times \hat{\mathbf{x}}$$

Finally, the components of the matrix that transforms coordinates from the EME2000 system to the Mars-centered mean equator and IAU node of epoch system are as follows:

$$\mathbf{M} = \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{p}}_{Mars} \end{bmatrix}$$

APPENDIX D

Near-optimal Earth-to-Mars Example

This appendix is a near-optimal Earth-to-Mars example taken from Table 3 of "On the Nature of Earth-Mars Porkchop Plots". It is a Type II trajectory that starts and ends near the line of nodes.

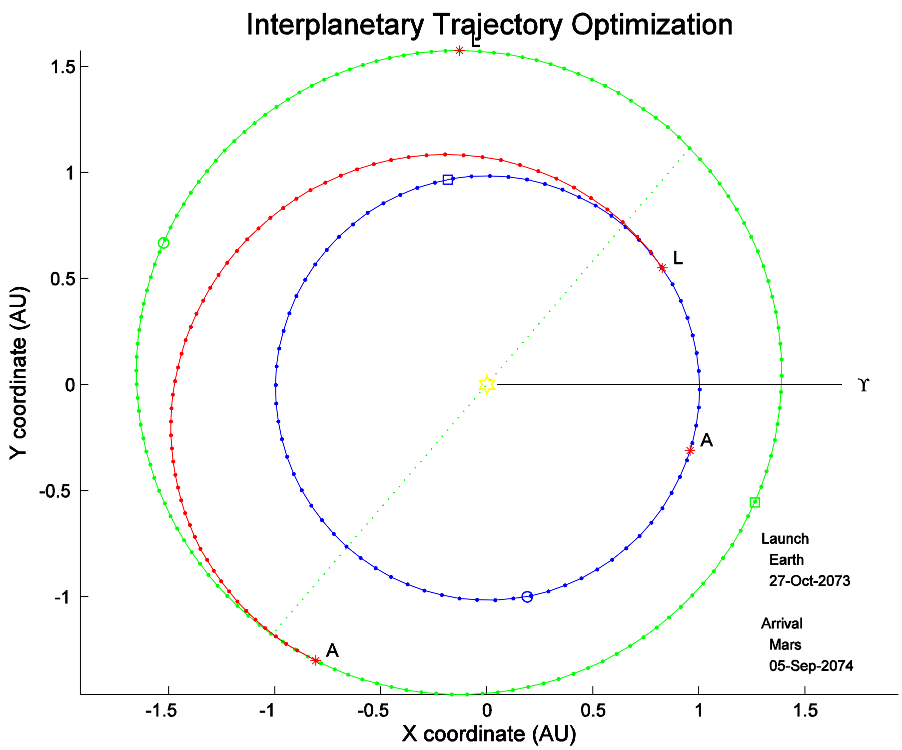
```
            program ipto_matlab

< interplanetary trajectory optimization >


departure conditions - start date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 10,15,2073

please input the departure date search boundary in days
? 30


arrival conditions - start date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 9,1,2074

please input the arrival date search boundary in days
? 30

 celestial body menu

  <1>  Mercury
  <2>  Venus
  <3>  Earth
  <4>  Mars
  <5>  Jupiter
  <6>  Saturn
  <7>  Uranus
  <8>  Neptune
  <9>  Pluto
  <10> asteroid/comet

please select the departure celestial body
? 3

 celestial body menu

  <1>  Mercury
  <2>  Venus
  <3>  Earth
  <4>  Mars
  <5>  Jupiter
  <6>  Saturn
  <7>  Uranus
  <8>  Neptune
  <9>  Pluto
  <10> asteroid/comet

please select the arrival celestial body
? 4


would you like to enforce mission constraints (y = yes, n = no)
? n

      optimization menu

 <1> minimize departure delta-v
```

```
 <2> minimize arrival delta-v

 <3> minimize total delta-v

 <4> no optimization

 selection (1, 2, 3 or 4)
? 3


       program ipto_matlab

     minimize total delta-v

departure celestial body     Earth

departure calendar date      27-Oct-2073
departure TDB time           09:45:45.752

departure julian date        2478507.9068


arrival celestial body       Mars

arrival calendar date        05-Sep-2074
arrival TDB time             07:06:59.387

arrival julian date          2478820.7965

transfer time                  312.8897  days

heliocentric orbital conditions prior to the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.4969408417e+08  1.6609453795e-02  1.1210898500e-02  2.8044520046e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 1.8037964394e+02  2.9274450652e+02  2.1318970698e+02  3.6560932598e+02

        rx (km)                 ry (km)                rz (km)                rmag (km)
 +1.23897648645804e+08  +8.22219010723092e+07  -1.59271354329400e+04  +1.48697910575695e+08

        vx (kps)                vy (kps)               vz (kps)               vmag (kps)
 -1.69521171087165e+01  +2.47196689743472e+01  -4.85869781455861e-03  +2.99739275417134e+01


heliocentric orbital conditions after the first maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.9096184451e+08  2.2162235455e-01  6.9290298183e-01  3.2334603687e+00

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 3.4076796787e+01  3.5625905616e+02  3.5949251652e+02  5.2678113071e+02

        rx (km)                 ry (km)                rz (km)                rmag (km)
 +1.23897648645804e+08  +8.22219010723092e+07  -1.59271354329400e+04  +1.48697910575695e+08

        vx (kps)                vy (kps)               vz (kps)               vmag (kps)
 -1.85788729046101e+01  +2.72890403856853e+01  +3.99259849140234e-01  +3.30155365245570e+01


heliocentric orbital conditions prior to the second maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.9096184451e+08  2.2162235455e-01  6.9290298183e-01  3.2334603687e+00

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 3.4076796787e+01  2.0083994844e+02  2.0407340881e+02  5.2678113071e+02
```

```
       rx (km)                ry (km)                rz (km)                rmag (km)
 -1.20854934202438e+08  -1.94529518984791e+08  -1.12971375762974e+06  +2.29017303125122e+08

       vx (kps)               vy (kps)               vz (kps)               vmag (kps)
 +1.93316889510978e+01  -9.49943866454759e+00  -2.26156867291437e-01  +2.15407678501051e+01


heliocentric orbital conditions after the second maneuver
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2793037367e+08  9.3573871355e-02  1.8435637601e+00  2.8709088031e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9332930572e+01  2.6172932802e+02  1.8882020833e+02  6.8693178575e+02

       rx (km)                ry (km)                rz (km)                rmag (km)
 -1.20854934202438e+08  -1.94529518984791e+08  -1.12971375762974e+06  +2.29017303125122e+08

       vx (kps)               vy (kps)               vz (kps)               vmag (kps)
 +2.14856494547696e+01  -1.07018196959563e+01  -7.49033288768392e-01  +2.40150604447643e+01


heliocentric orbital conditions of arrival body
(mean ecliptic and equinox of J2000)
------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2793037367e+08  9.3573871355e-02  1.8435637601e+00  2.8709088031e+02

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9332930572e+01  2.6172932802e+02  1.8882020833e+02  6.8693178575e+02

       rx (km)                ry (km)                rz (km)                rmag (km)
 -1.20854934202449e+08  -1.94529518984786e+08  -1.12971375762962e+06  +2.29017303125124e+08

       vx (kps)               vy (kps)               vz (kps)               vmag (kps)
 +2.14856494547688e+01  -1.07018196959575e+01  -7.49033288768398e-01  +2.40150604447641e+01


departure delta-v and energy requirements
(mean equator and equinox of J2000)
-----------------------------------

x-component of delta-v      -1626.755796  meters/second
y-component of delta-v       2196.603054  meters/second
z-component of delta-v       1392.808770  meters/second

delta-v magnitude            3067.786770  meters/second

energy                          9.411316  kilometers^2/second^2

asymptote right ascension     126.522832  degrees
asymptote declination          27.001315  degrees


arrival delta-v and energy requirements
(mean equator and equinox of J2000)
-----------------------------------

x-component of delta-v      -2153.960504  meters/second
y-component of delta-v        895.174732  meters/second
z-component of delta-v        958.009444  meters/second

delta-v magnitude            2521.639496  meters/second

energy                          6.358666  kilometers^2/second^2

Mars-mean-equator and IAU node of epoch

asymptote right ascension     108.777442  degrees
asymptote declination         -12.875232  degrees
```

```
total delta-v                5589.426267  meters/second

total energy                   15.769981  kilometers^2/second^2
```

Here's the heliocentric graphics display for this example.



The following plots of the primer vector and its derivative illustrate that this interplanetary transfer trajectory is nearly optimal according to primer vector theory.